

Optimal Error Estimates of Conservative Local Discontinuous Galerkin Method for Nonlinear Schrödinger Equation

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ABSTRACT. In this paper, we propose a conservative local discontinuous Galerkin method for one-dimensional nonlinear Schrödinger equation. By using special upwind-biased numerical fluxes, we establish the optimal rate of convergence $\mathcal{O}(h^{k+1})$, with polynomial of degree k and grid size h . Meanwhile, we show that this method preserves the charge conservation law and thus we call it a conservative local discontinuous Galerkin method. Numerical experiments verify our theoretical result.

1. Introduction

In this paper, we present a conservative local discontinuous Galerkin (CLDG) method with alternative numerical fluxes for nonlinear Schrödinger equation (NLS) equation

$$(1.1) \quad iu_t + u_{xx} + f(|u|^2)u = 0 \quad \text{in } [0, T] \times (x_l, x_r),$$

with sufficiently smooth initial data $u(0, x) = u_0(x)$, $x \in (x_l, x_r)$. We will mainly focus on the periodic boundary condition $u(t, x_l) = u(t, x_r)$, $t \in [0, T]$. Here $f(u)$ is an arbitrary smooth, real-valued function. Under the aforementioned boundary condition and assumption on f , Eq. (1.1) has the charge conservation law, i.e.,

$$\|u(t)\|^2 = \|u_0\|^2, \quad \forall t \in (0, T].$$

Our main observation is that the proposed LDG method preserves the charge and possesses an optimal convergence rate.

The discontinuous Galerkin (DG) method is a class of finite element methods using discontinuous, piecewise polynomials as the solution and the test spaces in the spatial direction. For a detailed description of the method as well as its implementation and applications, we refer the readers to the review paper [3]. The LDG method is an extension of the DG method aimed at solving partial differential equations (PDEs) containing higher than first order spatial derivatives. The idea of the LDG method is to rewrite the equations with higher order derivatives into a first order system, then apply the DG method on the system. The design of the numerical fluxes is the key ingredient to ensure stability. The LDG techniques

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have been developed for various high order PDEs, including convection diffusion equations [2] and nonlinear one-dimensional and two-dimensional KdV-type equations [10, 12]. More details about the LDG methods for high order time-dependent PDEs can be found in the review paper [10].

Since the basis functions can be completely discontinuous, the LDG methods have certain flexibility and advantage. It can be easily designed for any order of accuracy. In fact, the order of accuracy can be locally determined in each cell, thus for efficient h - p adaptivity. It is easy to handle complicated geometry and boundary conditions. It can be used on arbitrary triangulations, even those with hanging nodes. It is extremely local in data communications. The evolution of the solution in each cell needs to communicate only with its immediate neighborhoods, regardless of the order of accuracy. The methods have excellent parallel efficiency. Finally, there is provable cell entropy inequality and L^2 stability, for arbitrary scalar equations in any spatial dimension and any triangulation, for any order of accuracy, without limiters.

Some recent attempts have been made to apply the DG discretization to solve the Schrödinger equation, see [5, 9, 13, 14] and references therein. In [9], Xu and Shu developed an LDG method to solve the generalized NLS equation. For linear Schrödinger equation, they obtained an error estimate of order $k + \frac{1}{2}$ for polynomials of degree k . In [5], Lu, Cai and Zhang presented an LDG method for solving one-dimensional linear Schrödinger equation so that the mass is preserved numerically. Zhang, Yu and Feng presented a mass preserving direct discontinuous Galerkin method in [13] for the one-dimensional coupled NLS equations, and in [14] for both one and two dimensional NLS equations. Particularly, in [14] the conservation property is verified, and further validated by some long time simulation results.

Compared with the status of optimal L^2 error estimates for LDG methods solving time-dependent diffusive PDEs, for example, the convection diffusion equations [2, 7], optimal L^2 error estimates for LDG methods solving high order time-dependent wave equations are much more elusive. The main technical difficulty is the lack of coercivity and hence the control on the auxiliary variables in the LDG method which are approximations to the derivatives of the solution and the lack of control on the interface boundary terms. When these issues are not addressed carefully, optimal L^2 error estimates could not be obtained. In [9, 12], a priori L^2 error estimates with suboptimal order $k + \frac{1}{2}$ for the LDG method with \mathbb{P}^k elements for the linearized KdV equations and the linearized Schrödinger equation in one spatial dimension were obtained. For high order linear wave equations, [11] proposed a general approach for proving optimal error estimates $k + 1$ by utilizing the LDG method and its time derivatives with different test functions and fully making use of the so-called Gauss-Radau projections. In [8], the authors developed an energy conserving LDG method for solving the second order linear wave equation and showed an optimal error estimate. Recently, Liu [4] presented an optimal L^2 -error estimate of the semi-discrete direct DG method for convection-diffusion equations by using some global projections satisfying interface conditions.

The aim of this paper is to obtain the optimal rate of convergence order $k + 1$ for the CLDG method with a particular numerical flux and a special projections on the auxiliary variables in the equations. The optimal order of error estimates hold not only for the solution itself but also for the auxiliary variables in the CLDG method approximating the various order derivatives of the solution. To our best knowledge,

this is the first successful optimal L^2 error estimates of the CLDG methods for such high order equations when not purely upwind numerical fluxes are considered.

The paper is organized as follows. In Section 2, we present some notations used later in the paper and present the CLDG method for the Eq. (1.1). In Section 3, we show the charge conservation law and optimal convergence order results. Numerical experiments confirming the optimality of our theoretical results are given in Section 4. Concluding remarks are given in Section 5.

2. CLDG Method for NLS Equation

In this section we introduce notations and definitions to be used later in the paper and propose a CLDG method for Eq. (1.1).

2.1. Basic Notations. For an integer $N \geq 1$, define $\mathbb{Z}_N := \{1, 2, \dots, N\}$. We denote by \mathcal{I}_h a resellation of $\mathcal{O} = (x_l, x_r)$ divided into N cells $\mathcal{O}_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, and denote by $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ its center, $j \in \mathbb{Z}_N$. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and $h = \max_{j \in \mathbb{Z}_N} h_j$. Assume that the mesh is quasi-uniform in the sense that there exists a positive constant γ such that $\gamma h \leq h_j$ for any $j \in \mathbb{Z}_N$.

We define a finite-element space consisting of piecewise polynomials

$$V_h^k := \{v \in L^2(\mathcal{O}) : v|_{\mathcal{O}_j} \in \mathbb{P}^k(\mathcal{O}_j), j \in \mathbb{Z}_N\}, \quad k \in \mathbb{N},$$

where $\mathbb{P}^k(\mathcal{O}_j)$ denotes the space of polynomials of the degree up to k in each cell \mathcal{O}_j . Note that functions in V_h^k are allowed to be discontinuous across element interfaces.

The solution of the numerical method is denoted by u_h which belongs to V_h^k . We denote by $(u_h)_{j+\frac{1}{2}}^-$ and $(u_h)_{j+\frac{1}{2}}^+$ the left and right limits of u_h at $x_{j+\frac{1}{2}}$, respectively.

For any integer $l \geq 0$, denote by $\|\cdot\|_{W_\infty^l(\mathcal{O}_j)}$ the standard Sobolev norm on \mathcal{O}_j . We also denote by $W_\infty^l(\mathcal{I}_h)$ the broken Sobolev space, endowed with the norm

$$\|u\|_{W_\infty^l(\mathcal{I}_h)} := \max_{j \in \mathbb{Z}_N} \|u\|_{W_\infty^l(\mathcal{O}_j)}.$$

2.2. CLDG Method. In order to construct the CLDG method, we rewrite (1.1) as a first-order system

$$(2.1) \quad \begin{aligned} i u_t + v_x + f(|u|^2)u &= 0, \\ v - u_x &= 0. \end{aligned}$$

The LDG method for solving (2.1) is defined as follows: find $u_h, v_h \in V_h^k$ such that for all test functions $p_h, q_h \in V_h^k$ and all $j \in \mathbb{Z}_N$,

$$(2.2) \quad \begin{aligned} i \int_{\mathcal{O}_j} (u_h)_t p_h dx - \int_{\mathcal{O}_j} v_h (p_h)_x dx + (\widehat{v}_h p_h^-)_{j+\frac{1}{2}} - (\widehat{v}_h p_h^+)_{j-\frac{1}{2}} + \int_{\mathcal{O}_j} f(|u_h|^2) u_h p_h dx &= 0, \\ \int_{\mathcal{O}_j} v_h q_h dx + \int_{\mathcal{O}_j} u_h (q_h)_x dx - (\widehat{u}_h q_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h q_h^+)_{j-\frac{1}{2}} &= 0. \end{aligned}$$

In this paper, instead of using the purely upwind flux, we adopt the so-called upwind-based flux. To be more specific, we choose

$$(2.3) \quad \begin{aligned} \widehat{u}_h &= \theta u_h^- + (1 - \theta) u_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N, \\ \widehat{v}_h &= (1 - \theta) v_h^- + \theta v_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N, \end{aligned}$$

where $\theta \in [0, 1]$.

We decompose the complex function $u(t, x)$ into its real and imaginary parts by writing

$$u(t, x) = r(t, x) + is(t, x)$$

with r and s being real-valued functions. Under the new notation, (1.1) can be written as

$$\begin{aligned} r_t + s_{xx} + f(r^2 + s^2)s &= 0, \\ s_t - r_{xx} + f(r^2 + s^2)r &= 0, \end{aligned}$$

which is equivalent to the first-order system

$$\begin{aligned} p - r_x &= 0, \\ r_t + p_x + f(r^2 + s^2)s &= 0, \\ q - s_x &= 0, \\ s_t - q_x + f(r^2 + s^2)r &= 0. \end{aligned}$$

The LDG method (2.2) becomes to find $r_h, p_h, s_h, q_h \in V_h^k$ such that for any $v_h, \omega_h, \alpha_h, \beta_h \in V_h^k$,

$$\begin{aligned} (2.4) \quad & \int_{\mathcal{O}_j} (r_h)_t v_h dx - \int_{\mathcal{O}_j} p_h (v_h)_x dx + (\hat{p}_h v_h^-)_{j+\frac{1}{2}} - (\hat{p}_h v_h^+)_{j-\frac{1}{2}} + \int_{\mathcal{O}_j} f(r_h^2 + s_h^2) s_h v_h dx = 0, \\ & \int_{\mathcal{O}_j} p_h \omega_h dx + \int_{\mathcal{O}_j} s_h (\omega_h)_x dx - (\hat{s}_h \omega_h^-)_{j+\frac{1}{2}} + (\hat{s}_h \omega_h^+)_{j-\frac{1}{2}} = 0, \\ & \int_{\mathcal{O}_j} (s_h)_t \alpha_h dx + \int_{\mathcal{O}_j} q_h (\alpha_h)_x dx - (\hat{q}_h \alpha_h^-)_{j+\frac{1}{2}} + (\hat{q}_h \alpha_h^+)_{j-\frac{1}{2}} - \int_{\mathcal{O}_j} f(r_h^2 + s_h^2) r_h \alpha_h dx = 0, \\ & \int_{\mathcal{O}_j} q_h \beta_h dx + \int_{\mathcal{O}_j} r_h (\beta_h)_x dx - (\hat{r}_h \beta_h^-)_{j+\frac{1}{2}} + (\hat{r}_h \beta_h^+)_{j-\frac{1}{2}} = 0, \end{aligned}$$

and the numerical fluxes become

$$\begin{aligned} (2.5) \quad & \hat{r}_h = \theta r_h^- + (1 - \theta) r_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N, \\ & \hat{p}_h = (1 - \theta) p_h^- + \theta p_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N, \\ & \hat{s}_h = \theta s_h^- + (1 - \theta) s_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N, \\ & \hat{q}_h = (1 - \theta) q_h^- + \theta q_h^+ \quad \text{at } x_{j+\frac{1}{2}}, \quad j = 0, 1, \dots, N. \end{aligned}$$

3. Main Results

3.1. Charge Conservation Law. In this subsection, we present the charge conservation law of LDG method (2.2).

PROPOSITION 3.1. *There exist numerical entropy fluxes $\hat{\phi}_{j+\frac{1}{2}}$ such that the solution to the method (2.2) and (2.3) satisfies*

$$(3.1) \quad \frac{d}{dt} \left[\int_{\mathcal{O}_j} |u_h|^2 dx \right] + \hat{\phi}_{j+\frac{1}{2}} - \hat{\phi}_{j-\frac{1}{2}} = 0.$$

PROOF. First, we take the complex conjugate for every term in (2.2)

$$(3.2) \quad \begin{aligned} -\mathbf{i} \int_{\mathcal{O}_j} (u_h^*)_t p_h^* dx - \int_{\mathcal{O}_j} v_h^* (p_h^*)_x dx + (\widehat{v}_h^* p_h^{*-})_{j+\frac{1}{2}} - (\widehat{v}_h^* p_h^{*+})_{j-\frac{1}{2}} + \int_{\mathcal{O}_j} f(|u|^2) u_h^* p_h^* dx &= 0, \\ \int_{\mathcal{O}_j} v_h^* q_h^* dx + \int_{\mathcal{O}_j} u_h^* (q_h^*)_x dx - (\widehat{u}_h^* q_h^{*-})_{j+\frac{1}{2}} + (\widehat{u}_h^* q_h^{*+})_{j-\frac{1}{2}} &= 0, \end{aligned}$$

where u_h^* denotes the complex conjugate of u_h . Since (2.2) and (3.2) hold for any test functions in V_h^k , we choose $p_h = u_h^*$, $q_h = v_h^*$. With these choices of test functions, it follows that

$$(3.3) \quad \begin{aligned} -\mathbf{i} \int_{\mathcal{O}_j} (u_h^*)_t u_h dx - \int_{\mathcal{O}_j} v_h^* (u_h)_x dx + (\widehat{v}_h^* u_h^-)_{j+\frac{1}{2}} - (\widehat{v}_h^* u_h^+)_{j-\frac{1}{2}} + \int_{\mathcal{O}_j} f(|u_h|^2) |u_h|^2 dx &= 0, \\ \int_{\mathcal{O}_j} v_h^* v_h dx + \int_{\mathcal{O}_j} u_h^* (v_h)_x dx - (\widehat{u}_h^* v_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h^* v_h^+)_{j-\frac{1}{2}} &= 0, \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} \mathbf{i} \int_{\mathcal{O}_j} (u_h)_t u_h^* dx - \int_{\mathcal{O}_j} v_h (u_h^*)_x dx + (\widehat{v}_h u_h^{*-})_{j+\frac{1}{2}} - (\widehat{v}_h u_h^{*+})_{j-\frac{1}{2}} + \int_{\mathcal{O}_j} f(|u_h|^2) |u_h|^2 dx &= 0, \\ \int_{\mathcal{O}_j} v_h v_h^* dx + \int_{\mathcal{O}_j} u_h (v_h^*)_x dx - (\widehat{u}_h v_h^{*-})_{j+\frac{1}{2}} + (\widehat{u}_h v_h^{*+})_{j-\frac{1}{2}} &= 0. \end{aligned}$$

Taking the difference between the sum of the two equalities in (3.3) and the sum of the two equalities in (3.4) leads to

$$(3.5) \quad \begin{aligned} \mathbf{i} \int_{\mathcal{O}_j} ((u_h)_t u_h^* + (u_h^*)_t u_h) dx - \int_{\mathcal{O}_j} (v_h (u_h^*)_x - v_h^* (u_h)_x - u_h (v_h^*)_x + u_h^* (v_h)_x) dx \\ - (\widehat{u}_h v_h^{*-})_{j+\frac{1}{2}} + (\widehat{u}_h^* v_h^-)_{j+\frac{1}{2}} - (\widehat{v}_h^* u_h^-)_{j+\frac{1}{2}} + (\widehat{v}_h u_h^{*-})_{j+\frac{1}{2}} \\ + (\widehat{u}_h v_h^{*+})_{j-\frac{1}{2}} - (\widehat{u}_h^* v_h^+)_{j-\frac{1}{2}} + (\widehat{v}_h^* u_h^+)_{j-\frac{1}{2}} - (\widehat{v}_h u_h^{*+})_{j-\frac{1}{2}} &= 0. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \int_{\mathcal{O}_j} (v_h (u_h^*)_x - v_h^* (u_h)_x - u_h (v_h^*)_x + u_h^* (v_h)_x) dx \\ = (u_h^- v_h^{*-})_{j+\frac{1}{2}} - (u_h^+ v_h^{*+})_{j-\frac{1}{2}} - (v_h^- u_h^{*-})_{j+\frac{1}{2}} + (v_h^+ u_h^{*+})_{j-\frac{1}{2}}. \end{aligned}$$

Substituting it into Eq. (3.5), we obtain

$$\frac{d}{dt} \left[\int_{\mathcal{O}_j} |u_h|^2 dx \right] + \hat{\phi}_{j+\frac{1}{2}} - \hat{\phi}_{j-\frac{1}{2}} = 0,$$

where the numerical entropy flux is given by

$$\hat{\phi} = 2\text{Im} (\theta v_h^+ u_h^{*-} + (1-\theta) v_h^- u_h^{*+}).$$

This completes the proof. \square

THEOREM 3.1. The solution to the LDG method (2.2) and (2.3) has the charge conservation law, i.e.,

$$(3.6) \quad \|u_h(t)\|^2 = \|u_0\|^2, \quad \forall t \in (0, T].$$

PROOF. Summing up Eq. (3.1) with j over $\{1, 2, \dots, N\}$ and using the periodic boundary condition, we have

$$\frac{d}{dt} \left[\int_0^1 |u_h|^2 dx \right] = 0,$$

from which we obtain (3.6). \square

REMARK 3.1. We call the LDG method (2.2)–(2.3) a CLDG method. The charge conservation law trivially implies an L^2 -stability of the numerical solution.

3.2. Optimal Error Estimates. In this subsection, we obtain the optimal error estimates for the approximations $r_h, s_h \in V_h^k$, which are given by the CLDG method (2.4).

3.2.1. *Projection and Interpolation Properties.* In what follows, we consider two special projections of a function ω with $k+1$ continuous derivatives into the space V_h^k . The special projections \mathcal{P} , \mathcal{Q} are defined as follows. Given a function $u \in H^1(\mathcal{I}_h)$ and any subinterval \mathcal{O}_j , it holds that

$$(3.7) \quad \int_{\mathcal{O}_j} (\mathcal{P}u(x) - u(x))\omega dx = 0 \quad \text{for all } \omega \in \mathbb{P}^{k-1}(\mathcal{O}_j) \quad \text{and} \quad \widehat{\mathcal{P}}u_{j+\frac{1}{2}} = \widehat{u}_{j+\frac{1}{2}},$$

and

$$(3.8) \quad \int_{\mathcal{O}_j} (\mathcal{Q}u(x) - u(x))\omega dx = 0 \quad \text{for all } \omega \in \mathbb{P}^k(\mathcal{O}_j) \quad \text{and} \quad \widehat{\mathcal{Q}}u_{j+\frac{1}{2}} = \widehat{u}_{j+\frac{1}{2}}.$$

Here and below, we denote $\hat{\omega} := \theta\omega^- + (1-\theta)\omega^+$ for any $\omega \in H^1(\mathcal{I}_h)$. In particular, when $\theta = 1$, \mathcal{P} , \mathcal{Q} are Gauss-Radau projection \mathcal{P}^- and \mathcal{Q}^- , respectively. It is well-known that (see e.g. [1], Theorem 3.1.6) there holds for any \mathcal{O}_j that

$$(3.9) \quad \begin{aligned} \|u - \mathcal{P}^-u\|_{L^\infty(\mathcal{O}_j)} &\leq Ch^{k+1}\|u\|_{W_\infty^{k+1}(\mathcal{O}_j)}, \\ \|u - \mathcal{Q}^-u\|_{L^\infty(\mathcal{O}_j)} &\leq Ch^{k+1}\|u\|_{W_\infty^{k+1}(\mathcal{O}_j)}. \end{aligned}$$

The projections mentioned above are well-defined. We also have the following estimates of these projections.

THEOREM 3.2. Assume that $u \in W_\infty^{k+1}(\mathcal{I}_h)$. For any $\theta \neq 1/2$, there exists $C = C(\theta)$ which is independent of h such that

$$(3.10) \quad \begin{aligned} \|u - \mathcal{P}u\|_{L^2(\mathcal{O})} &\leq Ch^{k+1}\|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}, \\ \|u - \mathcal{Q}u\|_{L^2(\mathcal{O})} &\leq Ch^{k+1}\|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}. \end{aligned}$$

PROOF. We begin the proof of (3.10) for \mathcal{P} . Denote by \mathcal{P}^- the Gauss-Radau projection and $E := \mathcal{P} - \mathcal{P}^-$. Since \mathcal{P}^- is unique, the existence and uniqueness of \mathcal{P} are equivalent to those of E . [6] has proved that E_j , the restriction of E to each \mathcal{O}_j , can be represented as

$$E_j(x) = \sum_{l=0}^k \alpha_{j,l} P_{j,l}(x) = \sum_{l=0}^k \alpha_{j,l} P_l(\xi),$$

where $P_l(\xi)$ are the l -order Legendre polynomials and are orthogonal on $[-1, 1]$ with $\xi = 2(x - x_j)/h_j$ and on each element, $P_{j,l}(x) := P_l(\xi)$ for $x \in \mathcal{O}_j$ and the coefficients satisfy

$$\alpha_{j,l} = 0, \quad l = 0, 1, \dots, k-1; \quad j = 1, \dots, N.$$

Moreover, if we define $\eta_{j+1} := (u - \mathcal{P}^- u)_{j+\frac{1}{2}}^+$ for $j = 0, 1, \dots, N-1$ with $\eta_{N+1} = \eta_1$, $\eta = (\eta_2, \eta_3, \dots, \eta_{N+1})^T$ and $\alpha_k = (\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{N,k})^T$, then

$$A\alpha_k = (1 - \theta)\eta,$$

where $A = \text{circ}(\theta, (1 - \theta)(-1)^k, 0, \dots, 0)$ is an $N \times N$ circulant matrix. The determinant of A is

$$|A| = \theta^N (1 - q^N) \quad \text{with} \quad q = \frac{(1 - \theta)(-1)^{k+1}}{\theta},$$

from which we conclude A is always invertible for all k and N whenever $\theta \neq 1/2$. This establishes existence and uniqueness of \mathcal{P} . When $\theta \neq 1/2$, the inverse of A is

$$A^{-1} = \beta_N \text{circ}(1, q, \dots, q^{N-1}) \quad \text{with} \quad \beta_N = \frac{1}{\theta(1 - q^N)}.$$

In this case,

$$\alpha_{j,k} = \beta_N (1 - \theta) \sum_{m=1}^N d_{j,m} \eta_{m+1}, \quad j = 1, \dots, N,$$

where $\{d_{j,m}\}_{m=1}^N$, $j = 1, \dots, N$, are the entries in the j -th row of the circulant matrix $\text{circ}(1, q, \dots, q^{N-1})$. By the first estimate in (3.9),

$$|\eta_{j+1}| \leq \|u - \mathcal{P}^- u\|_{L^\infty(\mathcal{O}_{j+1})} \leq Ch^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{O}_{j+1})} \leq Ch^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)},$$

from which we get a bound for $\alpha_{j,k}$:

$$\begin{aligned} |\alpha_{j,k}| &\leq C|\beta_N| |1 - \theta| h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)} (1 + |q| + \dots + |q|^{N-1}) \\ &= \frac{C(1 - \theta)}{1 - 2\theta} h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)} := C(\theta) h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}. \end{aligned}$$

This yields

$$\|E\|_{L^\infty(\mathcal{O}_j)} = |\alpha_{j,k}| \|P_k\|_{L^\infty([-1,1])} \leq C(\theta) h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}.$$

As a consequence,

$$\|u - \mathcal{P}u\|_{L^\infty(\mathcal{O}_j)} \leq \|u - \mathcal{P}^- u\|_{L^\infty(\mathcal{O}_j)} + \|E\|_{L^\infty(\mathcal{O}_j)} \leq C(\theta) h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}.$$

Therefore,

$$\|u - \mathcal{P}u\|_{L^2(\mathcal{O}_j)} \leq C(\theta) h^{k+\frac{3}{2}} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}$$

and

$$\|u - \mathcal{P}u\|_{L^2(\mathcal{O})} \leq C(\theta) h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}.$$

This proves the second estimate of (3.10). Similar arguments combined with the second estimate in (3.9) shows (3.10) for \mathcal{Q} . \square

REMARK 3.2. When $\theta = 1/2$, since the determinant of A is

$$|A| = \theta^N - (1 - \theta)^N (-1)^{N(k+1)} = 2^{-N} \left(1 - (-1)^{N(k+1)}\right),$$

A is inverse if and only if N is odd and k is even. In this case, $q = -1$ and $\beta_N = 1$. Then

$$|\alpha_{j,k}| \leq CN h^{k+1} \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)} \leq Ch^k \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)},$$

and thus

$$\begin{aligned}\|u - \mathcal{P}u\|_{L^2(\mathcal{O})} &\leq Ch^k \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}, \\ \|u - \mathcal{Q}u\|_{L^2(\mathcal{O})} &\leq Ch^k \|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}.\end{aligned}$$

Compared with the estimate (3.2) for $\theta \neq 1/2$, the order of projection for $\theta = 1/2$ reduces one.

3.2.2. Notations for the CLDG Discretization. To facilitate the proof of the error estimate, we define the CLDG discretization operator \mathcal{B} , i.e., for each subinterval \mathcal{O}_j ,

$$\begin{aligned}\mathcal{B}_j(r, p, s, q; v, \omega, \alpha, \beta) &= - \int_{\mathcal{O}_j} p v_x dx + \int_{\mathcal{O}_j} s \omega_x dx + \int_{\mathcal{O}_j} q \alpha_x dx + \int_{\mathcal{O}_j} r \beta_x dx \\ &\quad + (\widehat{p}v^-)_{j+\frac{1}{2}} - (\widehat{p}v^+)_{j-\frac{1}{2}} - (\widehat{s}\omega^-)_{j+\frac{1}{2}} + (\widehat{s}\omega^+)_{j-\frac{1}{2}} \\ &\quad - (\widehat{q}\alpha^-)_{j+\frac{1}{2}} + (\widehat{q}\alpha^+)_{j-\frac{1}{2}} - (\widehat{r}\beta^-)_{j+\frac{1}{2}} + (\widehat{r}\beta^+)_{j-\frac{1}{2}},\end{aligned}$$

and

$$\mathcal{B}(r, p, s, q; v, \omega, \alpha, \beta) := \sum_j \mathcal{B}_j(r, p, s, q; v, \omega, \alpha, \beta).$$

According to the periodic boundary condition and the definitions of the operator \mathcal{B} and the projections \mathcal{P} , \mathcal{Q} , we have

LEMMA 3.1. *For any $v_h, \omega_h, \alpha_h, \beta_h \in V_h^k$, it holds*

$$(3.11) \quad \mathcal{B}(r - \mathcal{P}r, p - \mathcal{Q}p, s - \mathcal{P}s, q - \mathcal{Q}q; v_h, \omega_h, \alpha_h, \beta_h) = 0,$$

where \mathcal{P} , \mathcal{Q} are the projections defined in section 3.2.1. \square

In addition, we define the LDG discretization operator \mathcal{H} for the nonlinear term, i.e., for each subinterval \mathcal{O}_j ,

$$\mathcal{H}_j(r, s; v, \alpha) := \int_{\mathcal{O}_j} f(r^2 + s^2) r \alpha dx - \int_{\mathcal{O}_j} f(r^2 + s^2) s v dx$$

and

$$\mathcal{H}(r, s; v, \alpha) := \sum_j \mathcal{H}_j(r, s; v, \alpha).$$

3.2.3. Optimal Error Estimates. With these notations and equalities (2.4), we have

$$\begin{aligned}\int_{\mathcal{O}_j} (r_h)_t v_h dx + \int_{\mathcal{O}_j} (s_h)_t \alpha_h dx + \int_{\mathcal{O}_j} p_h \omega_h dx + \int_{\mathcal{O}_j} q_h \beta_h dx \\ + \mathcal{B}_j(r_h, p_h, s_h, q_h; v_h, \omega_h, \alpha_h, \beta_h) - \mathcal{H}_j(r_h, s_h; v_h, \alpha_h) = 0.\end{aligned}$$

Notice that the CLDG method (2.4)–(2.5) is also satisfied when the numerical solutions r_h, p_h, s_h, q_h are replaced by the exact solutions $r, p = r_x, s, q = s_x$. This yields the following cell error equation:

$$\begin{aligned}\int_{\mathcal{O}_j} (r - r_h)_t v_h dx + \int_{\mathcal{O}_j} (s - s_h)_t \alpha_h dx + \int_{\mathcal{O}_j} (p - p_h) \omega_h dx \\ + \int_{\mathcal{O}_j} (q - q_h) \beta_h dx + \mathcal{B}_j(r - r_h, p - p_h, s - s_h, q - q_h; v_h, \omega_h, \alpha_h, \beta_h) \\ - \mathcal{H}_j(r, s; v_h, \alpha_h) + \mathcal{H}_j(r_h, s_h; v_h, \alpha_h) = 0, \quad v_h, \omega_h, \alpha_h, \beta_h \in V_h^k.\end{aligned}$$

Summing over j , we get

$$\begin{aligned} & \int_{\mathcal{O}} (r - r_h)_t v_h dx + \int_{\mathcal{O}} (s - s_h)_t \alpha_h dx + \int_{\mathcal{O}} (p - p_h) \omega_h dx \\ & + \int_{\mathcal{O}} (q - q_h) \beta_h dx + \mathcal{B}(r - r_h, p - p_h, s - s_h, q - q_h; v_h, \omega_h, \alpha_h, \beta_h) \\ & = \mathcal{H}(r, s; v_h, \alpha_h) - \mathcal{H}(r_h, s_h; v_h, \alpha_h), \quad v_h, \omega_h, \alpha_h, \beta_h \in V_h^k. \end{aligned}$$

Denoting

$$\begin{aligned} e_r &= r - r_h = r - \mathcal{P}r + \mathcal{P}e_r, \\ e_s &= s - s_h = s - \mathcal{P}s + \mathcal{P}e_s, \\ e_p &= p - p_h = p - \mathcal{Q}p + \mathcal{Q}e_p, \\ e_q &= q - q_h = q - \mathcal{Q}q + \mathcal{Q}e_q, \end{aligned}$$

and taking the test functions

$$v_h = \mathcal{P}r - r_h, \quad \omega_h = \mathcal{Q}p - p_h, \quad \alpha_h = \mathcal{P}s - s_h, \quad \beta_h = \mathcal{Q}q - q_h,$$

we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int_{\mathcal{O}} (\mathcal{P}e_r)^2 dx \right] + \frac{1}{2} \frac{d}{dt} \left[\int_{\mathcal{O}} (\mathcal{P}e_s)^2 dx \right] + \int_{\mathcal{O}} (r - \mathcal{P}r)_t \mathcal{P}e_r dx \\ & + \int_{\mathcal{O}} (s - \mathcal{P}s)_t \mathcal{P}e_s dx + \int_{\mathcal{O}} (p - p_h) \mathcal{Q}e_p dx + \int_{\mathcal{O}} (q - q_h) \mathcal{Q}e_q dx \\ & + \mathcal{B}(r - \mathcal{P}r, p - \mathcal{Q}p, s - \mathcal{P}s, q - \mathcal{Q}q; \mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q) \\ & + \mathcal{B}(\mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q; \mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q) \\ & = \mathcal{H}(r, s; \mathcal{P}e_r, \mathcal{P}e_s) - \mathcal{H}(r_h, s_h; \mathcal{P}e_r, \mathcal{P}e_s). \end{aligned}$$

It follows from (3.11) in Lemma 3.1 that

$$\mathcal{B}(r - \mathcal{P}r, p - \mathcal{Q}p, s - \mathcal{P}s, q - \mathcal{Q}q; \mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q) = 0.$$

By the same argument as that used for the charge conservation law,

$$\mathcal{B}(\mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q; \mathcal{P}e_r, \mathcal{Q}e_p, \mathcal{P}e_s, \mathcal{Q}e_q) = 0.$$

The above two equations imply

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int_{\mathcal{O}} (\mathcal{P}e_r)^2 dx \right] + \frac{1}{2} \frac{d}{dt} \left[\int_{\mathcal{O}} (\mathcal{P}e_s)^2 dx \right] + \int_{\mathcal{O}} (r - \mathcal{P}r)_t \mathcal{P}e_r dx \\ & + \int_{\mathcal{O}} (s - \mathcal{P}s)_t \mathcal{P}e_s dx + \int_{\mathcal{O}} (p - p_h) \mathcal{Q}e_p dx + \int_{\mathcal{O}} (q - q_h) \mathcal{Q}e_q dx \\ & = \mathcal{H}(r, s; \mathcal{P}e_r, \mathcal{P}e_s) - \mathcal{H}(r_h, s_h; \mathcal{P}e_r, \mathcal{P}e_s). \end{aligned}$$

An application of the Cauchy-Schwarz inequality and Young inequality gives

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \frac{d}{dt} \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2 + \|\mathcal{Q}e_p\|_{L^2(\mathcal{O})}^2 + \|\mathcal{Q}e_q\|_{L^2(\mathcal{O})}^2 \\ & \leq \|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2 + \|(r - \mathcal{P}r)_t\|_{L^2(\mathcal{O})}^2 + \|(s - \mathcal{P}s)_t\|_{L^2(\mathcal{O})}^2 \\ & \quad + \|p - \mathcal{Q}p\|_{L^2(\mathcal{O})}^2 + \|q - \mathcal{Q}q\|_{L^2(\mathcal{O})}^2 + 2I, \end{aligned}$$

with

$$\begin{aligned}
I &= |\mathcal{H}(r, s; \mathcal{P}e_r, \mathcal{P}e_s) - \mathcal{H}(r_h, s_h; \mathcal{P}e_r, \mathcal{P}e_s)| \\
&\leq \int_{\mathcal{O}} |f(r^2 + s^2)r - f(r_h^2 + s_h^2)r_h| \cdot |\mathcal{P}e_s| dx \\
&\quad + \int_{\mathcal{O}} |f(r^2 + s^2)s - f(r_h^2 + s_h^2)s_h| \cdot |\mathcal{P}e_r| dx =: I_1 + I_2.
\end{aligned}$$

Now, we give the error estimates of I_1 and I_2 , respectively.

Assume $f(u) \in C^1(\mathcal{O})$ and set $g(r, s) = f(r^2 + s^2)r$. The Taylor expansion yields

$$\begin{aligned}
I_1 &\leq \int_{\mathcal{O}} |(g_1)'_r| |e_r \mathcal{P}e_s| dx + \int_{\mathcal{O}} |(g_1)'_s| |e_s \mathcal{P}e_s| dx \\
&\leq C_1 \left(\|s - \mathcal{P}s\|_{L^2(\mathcal{O})}^2 + \|r - \mathcal{P}r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2 \right).
\end{aligned}$$

In the similar manner, I_2 has analogous estimate. Thus

$$I \leq C_2 \left(\|s - \mathcal{P}s\|_{L^2(\mathcal{O})}^2 + \|r - \mathcal{P}r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2 \right).$$

Applying the interpolation properties (3.10) in Theorem 3.2, we obtain

$$\begin{aligned}
\frac{d}{dt} \|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \frac{d}{dt} \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2 &\leq C_3 (\|\mathcal{P}e_r\|_{L^2(\mathcal{O})}^2 + \|\mathcal{P}e_s\|_{L^2(\mathcal{O})}^2) + C_4 h^{2k+2} \\
&\quad \left(\|r\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 + \|s\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 + \|r_t\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 \right. \\
&\quad \left. + \|s_t\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 + \|p\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 + \|q\|_{W_\infty^{k+1}(\mathcal{I}_h)}^2 \right).
\end{aligned}$$

Finally, we conclude by Gronwall inequality that

$$(3.12) \quad \|\mathcal{P}e_r\|_{L^2(\mathcal{O})} + \|\mathcal{P}e_s\|_{L^2(\mathcal{O})} \leq C_5 h^{k+1}.$$

Combining (3.10) and (3.12), we obtain the desired optimal error estimates.

THEOREM 3.3. Let u and u_h be the numerical solution of Eq. (1.1) and Eq. (2.4), respectively, with $\theta \neq 1/2$. Assume that $f \in C^1$ and that u is sufficiently smooth with bounded derivatives. For regular triangulations of \mathcal{O} , if the finite element space V_h^k is the piecewise polynomials of degree $k \geq 1$, then for small enough h there holds the following error estimates

$$\|u - u_h\|_{L^2(\mathcal{O})} \leq Ch^{k+1},$$

where the constant C depends only on the final time T , θ , $\|u\|_{W_\infty^{k+1}(\mathcal{I}_h)}$, $\|u_t\|_{W_\infty^{k+1}(\mathcal{I}_h)}$, $\|u_x\|_{W_\infty^{k+1}(\mathcal{I}_h)}$ and $|f'|$, but not on the mesh size h .

REMARK 3.3. Even though the proof in this section is presented only for simple one-dimensional system, the same optimal convergence results can be obtained for high-dimensional case.

4. Numerical Experiments

In this section, we will present some detailed numerical investigations of the CLDG method (2.4)–(2.5) to the following NLS equation

$$(4.1) \quad \mathbf{i}u_t + u_{xx} + 2|u|^2u = 0.$$

Time discretization is by the implicit midpoint scheme. In particular, we will focus on the charge conservation law and accuracy of the method.

4.1. The evolution of single soliton. Consider Eq. (4.1) with a single soliton solution

$$u(t, x) = \text{sech}(x + x_0 - 4t)e^{2i(x+x_0-3t/2)}.$$

In the following experiments, we take the temporal step-size $\tau = 0.001$, the spatial meshgrid-size $h = 0.5$, and the time interval $[0, 5]$, the numerical spatial domain $\mathcal{O} = [-25, 25]$ with the periodic boundary condition.

The intensity profiles of the exact solution and numerical solution are shown in Fig. 1. We observe a very good behavior of our method and a good agreement with the theoretical solution.

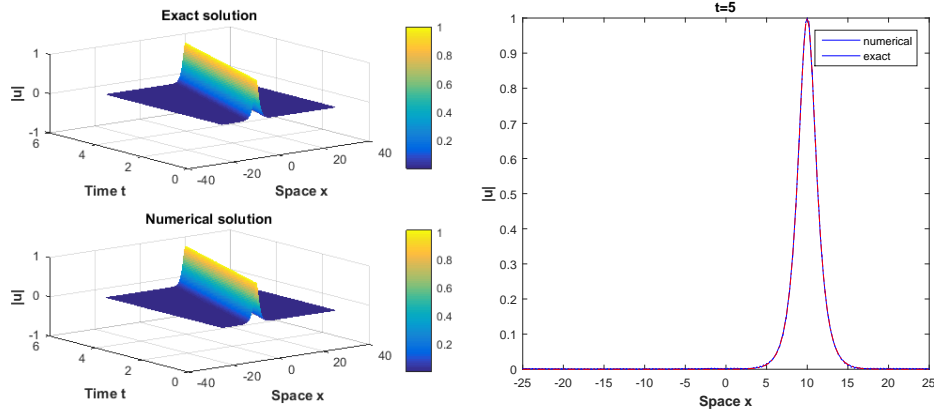


FIGURE 1. Comparison of the analytic solution and the numerical solution with $x_0 = 10$ and $\theta = 1$.

As is stated in Theorem 3.1, the LDG method (2.2)-(2.3) could preserve the discrete charge conservation law exactly. We consider this phenomenon numerically in Fig. 2, where the figure shows the global error. We can see that the global residual of the discrete charge conservation law reaches the magnitude of 10^{-15} . Thus, we observe a good agreement with the theoretical result.

4.2. The Interaction of Double Soliton. In this experiment, we show the double soliton collision of Eq. (4.1) with the initial condition

$$u_0(x) = \text{sech}(x - x_1)e^{2ic_1(x-x_1)} + \text{sech}(x - x_2)e^{2ic_2(x-x_2)}.$$

The global error of the discrete charge conservation law is shown in Fig. 3. Again, we observe the phenomena which agrees with the theoretical result.

The evolution on the interval $[0; 5]$ is shown in Fig. 4 and the profiles at different instants in Fig. 5. We observe that the interaction is elastic and the two waves emerges without any changes in their shapes and they conserve the energy almost exactly.

4.3. The Birth of Mobile Soliton. In this experiment, we show the birth of soliton using a square well initial condition

$$u_0(x) = Ae^{-x^2+2ix}.$$

In Figs. 6-7, the mobile soliton is observed.

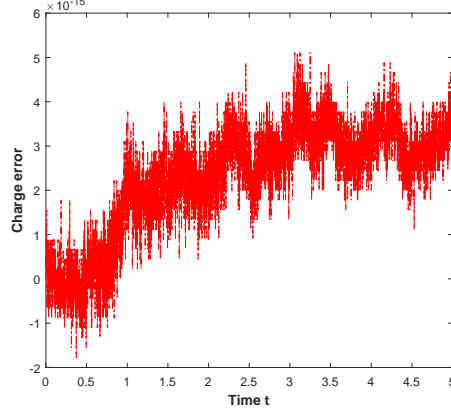


FIGURE 2. The global error of the discrete charge conservation law of single soliton with $x_0 = 10$ and $\theta = 1$.

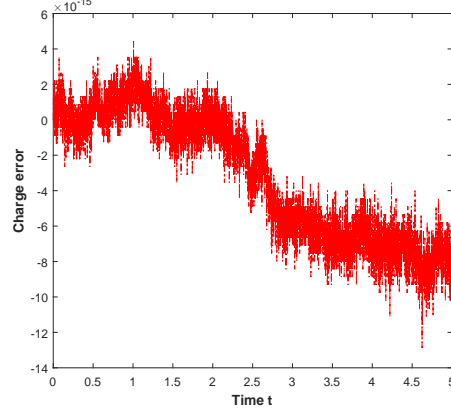


FIGURE 3. The global error of the discrete charge conservation law of double soliton with $c_1 = 1$, $c_2 = 1$, $x_1 = -10$, $x_2 = 10$ and $\theta = 1$.

4.4. Optimal Convergence Estimates. We show an accuracy test for Eq. (4.1) with the soliton solution

$$u(t, x) = \text{sech}(x + x_0 - 4t)e^{2i(x + x_0 - 3t/2)}.$$

We take the temporal step-size $\tau = 0.00001$, and the time interval $[0, 1]$, the numerical spatial domain $\mathcal{O} = [-30, 30]$ with the periodic boundary condition.

Table 1 lists the L^2 errors and their numerical orders with different values of $\theta \in [0, 1]$ at $T = 1$. From the table we conclude that, for all values of $\theta > 0$, one can always observe $(k + 1)$ -th order of accuracy in L^2 -norm.

5. Conclusion

In this paper, we develop a LDG method to solve one-dimensional Schrödinger equation subject to nonlinear potential. The charge conservation law is shown

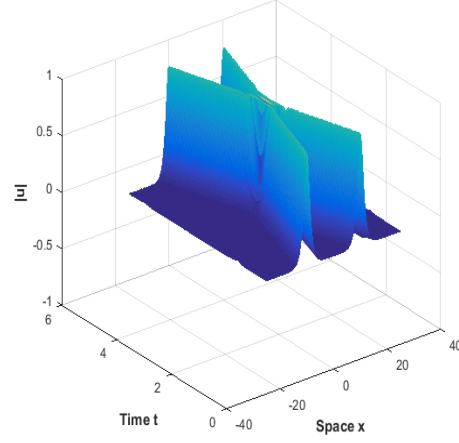


FIGURE 4. Collision of two solitons whose parameters are $c_1 = 1, c_2 = 1, x_1 = -10, x_2 = 10, \theta = 1$.

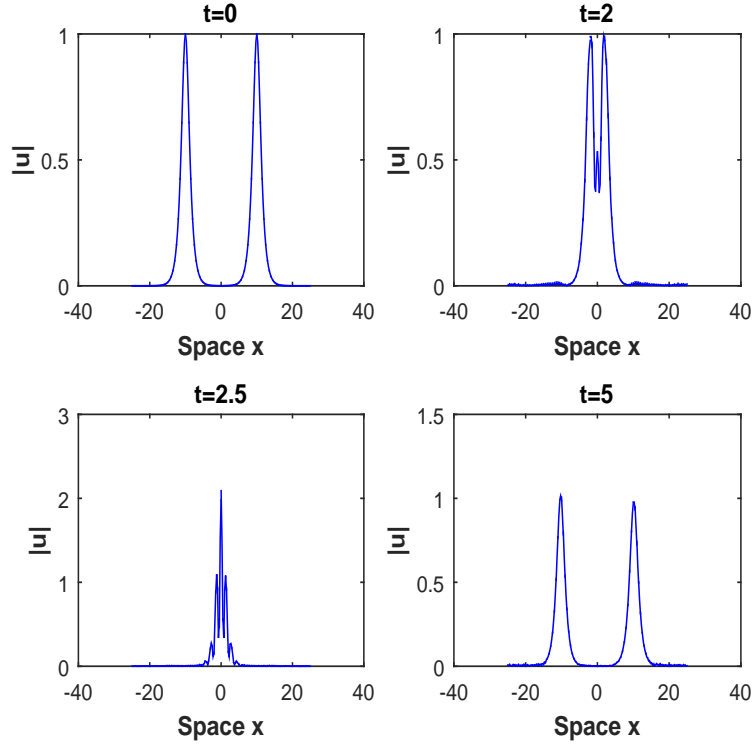


FIGURE 5. Profiles at time $t = 0, 2, 2.5$ and 5 of Fig. 4.

to be preserved for LDG method proposed in this paper. The CLDG method, when applied to NLS equation, is shown to have the optimal $(k + 1)$ -th order of

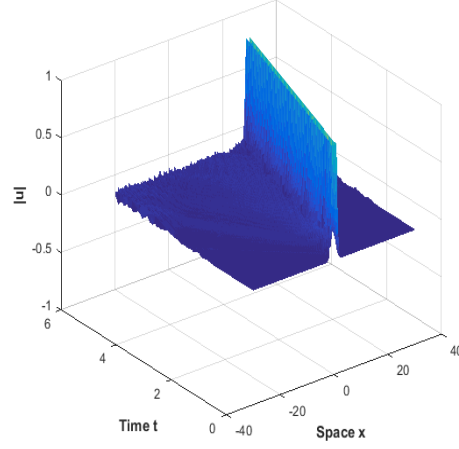


FIGURE 6. The birth of mobile soliton. $A = 2, \theta = 1$. Periodic boundary condition in $[-30, 30]$.

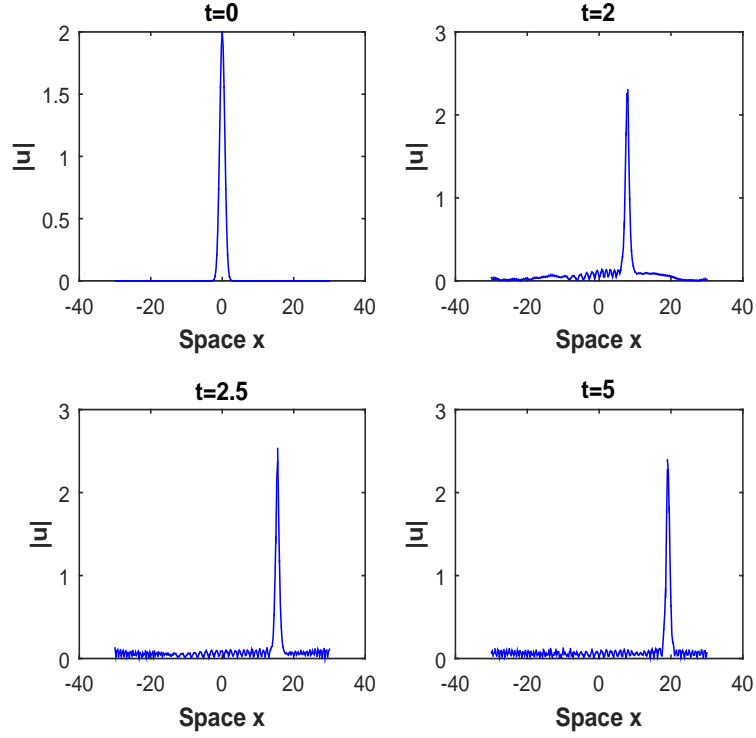


FIGURE 7. Profiles at time $t = 0, 2, 2.5$ and 5 of Fig. 6.

accuracy for polynomial elements of degree k . The numerical tests demonstrate both accuracy and capacity of the method.

	N	L^2 error	Order		N	L^2 error	Order
$\theta = 0.4$	60	3.54E-1	-	$\theta = 0.4$	60	8.72E-2	-
	120	7.20E-2	2.70		120	6.58E-3	3.94
	240	1.36E-2	3.02		240	5.64E-4	3.90
	480	2.43E-3	3.04		480	4.82E-5	3.91
$\theta = 0.5$	60	2.31E-1	-	$\theta = 0.5$	60	1.94E-2	-
	120	2.33E-2	3.73		120	2.18E-3	3.50
	240	3.89E-3	3.45		240	2.02E-4	3.87
	480	6.91E-4	2.93		480	1.84E-5	3.80
$\theta = 1$	60	2.30E-1	-	$\theta = 1$	60	2.91E-2	-
	120	2.89E-2	3.24		120	2.99E-3	3.84
	240	4.54E-3	3.68		240	2.64E-4	3.80
	480	7.51E-4	2.94		480	2.67E-5	3.97

TABLE 1. Accuracy test for Eq. (1.1) using \mathbb{P}^2 polynomials (left) and \mathbb{P}^3 polynomials (right) with different θ .

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